Sparse Matrix Methods and Probabilistic Inference Algorithms

Radford M. Neal

Dept. of Statistics and Dept. of Computer Science
University of Toronto

http://www.cs.utoronto.ca/~radford/
radford@stat.utoronto.ca

Part I

Faster Encoding for Low Density Parity Check Codes Using Sparse Matrix Methods

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The Parity Check Matrix

Suppose we will send blocks of N bits (0's and 1's) through a channel.

To be able to correct errors, we reduce the number of possible blocks by requiring that a block satisfy M parity checks.

We can express this by saying a valid block (or *codeword*) must satisfy

$$\mathbf{Hx} = \mathbf{0}$$

Here $\mathbf{x}$, the codeword, is a column vector of N bits, $\mathbf{0}$ is a column vector of N zeros, and $\mathbf{H}$ is an $M \times N$ *parity check matrix*, with $M < N$.

All arithmetic is done modulo 2 (equivalently, in GF(2)), where addition and subtraction are both XOR, and multiplication is AND.

The Encoding Problem

Let us assume that the rows of $\mathbf{H}$ are linearly independent. There will then be 2^{N-M} valid codewords, and we can use a codeword to uniquely represent a source block of $N-M$ bits.

The encoding problem: Define and compute a mapping from these $N-M$ source bits to the N bits of a codeword.

We will consider only *systematic* mappings, in which the $N-M$ source bits are directly represented by a subset of the N codeword bits. (The receiver can then easily find them.)

The other M bits of the codeword are chosen to satisfy the parity checks. We need to:

1) Choose which are the systematic source bits, and which are the parity check bits.

2) Figure out how to compute the M parity check bits given the $N-M$ source bits.

A Dense Encoding Method

Let's partition $\mathbf{H}$ into an $M \times M$ left part, $\mathbf{A}$, and an $M \times N$ right part, $\mathbf{B}$, after rearranging columns if necessary to make $\mathbf{A}$ non-singular.

Partition a codeword, $\mathbf{x}$, in the same way, into M check bits, $\mathbf{c}$, and $N-M$ source bits, $\mathbf{s}$.

The parity check equation, $\mathbf{Hx} = \mathbf{0}$, becomes

$$[\ \mathbf{A} \mid \mathbf{B}\ ] \left[\begin{array}{c} \mathbf{c} \\ - \\ \mathbf{s} \end{array} \right] = \mathbf{0}$$

From this, we get

$$\mathbf{Ac} + \mathbf{Bs} = \mathbf{0}$$

and hence

$$\mathbf{c} = \mathbf{A}^{-1}\mathbf{Bs}$$

We can pre-compute $\mathbf{A}^{-1}\mathbf{B}$, and then find the check bits $\mathbf{c}$ by multiplying the source bits $\mathbf{s}$ by this matrix. This takes time proportional to $M(N-M)$.

A Mixed Encoding Method

Suppose $\mathbf{H} = [\ \mathbf{A} \mid \mathbf{B}\ ]$ is sparse, and hence that $\mathbf{B}$ is as well. For LDPC codes, the number of 1's in a row of $\mathbf{B}$ will be constant, at least on average, independent of N.

It may then be faster to compute $\mathbf{c} = \mathbf{A}^{-1}\mathbf{B}\mathbf{s}$ in two steps:

1) Compute $\mathbf{z} = \mathbf{B}\mathbf{s}$, in time proportional to M, exploiting the sparseness of $\mathbf{B}$.

2) Compute $\mathbf{c} = \mathbf{A}^{-1}\mathbf{z}$, in time proportional to M^2.

The total time is of order M^2. This is better than the previous order $M(N-M)$ method when $M < N-M$ — ie, when the rate of the code is greater than 1/2.

We will next see how sparsity in $\mathbf{A}$ can be exploited as well.

Reduction to Upper Triangular Form

We can find $\mathbf{c} = \mathbf{A}^{-1}\mathbf{z}$ by using row operations to reduce $\mathbf{A}$ to an upper triangular matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ \underline{0} & 1 & 1 & \underline{0} \\ 0 & 0 & 1 & 0 \\ \underline{0} & 1 & 0 & \underline{1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \underline{0} & \underline{1} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \underline{1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \underline{0} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Using *backward substitution*, we can now find that $c_4 = 1$, $c_3 = 0$, $c_2 = 1$, $c_1 = 1$.

Recording the Reductions in a Lower Triangular Matrix

The previous process reduced the equation $\mathbf{Ac} = \mathbf{z}$ to $\mathbf{Uc} = \mathbf{y}$, where $\mathbf{U}$ is upper triangular, and $\mathbf{y}$ was found as we reduced $\mathbf{A}$ to $\mathbf{U}$.

To solve $\mathbf{Ac} = \mathbf{z}$ for many $\mathbf{z}$ without going through the reduction process every time, we record how to find $\mathbf{y}$ as the solution of $\mathbf{Ly} = \mathbf{z}$, where $\mathbf{L}$ is lower triangular. This equation is easily solved by *forward substitution*.

For the example, we get:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

which can be solved to give $y_1 = 0$, $y_2 = 1$, $y_3 = 0$, $y_4 = 1$.

Putting it All Together

For the reduction to work, $\mathbf{A}$ must be non-singular, with rows and columns ordered to give 1's on the diagonal when needed.

We can find such a sub-matrix as follows:

Set $\mathbf{U}$ and $\mathbf{L}$ to all zeros.
Set $\mathbf{F}$ to $\mathbf{H}$.
for $i = 1$ to M:
  Find a non-zero element of $\mathbf{F}$ that is in row i, column i, or in a later row/column.
  Rearrange rows and columns of $\mathbf{F}$ and $\mathbf{H}$ from i onward to put this element in row i, column i.
  Copy column i of $\mathbf{F}$ up to row i to column i of $\mathbf{U}$.
  Copy column i of $\mathbf{F}$ from row i to column i of $\mathbf{L}$.
  Add row i of $\mathbf{F}$ to later rows with a 1 in column i.
end
Set $\mathbf{B}$ to the last $N-M$ columns of the rearranged $\mathbf{H}$.

We use $\mathbf{B}$, $\mathbf{L}$, and $\mathbf{U}$ to find parity checks for $\mathbf{s}$:

Compute $\mathbf{z} = \mathbf{B}\mathbf{s}$, exploiting the sparseness of $\mathbf{B}$.
Solve $\mathbf{L}\mathbf{y} = \mathbf{z}$ for $\mathbf{y}$ by forward substitution.
Solve $\mathbf{U}\mathbf{c} = \mathbf{y}$ for $\mathbf{c}$ by backward substitution.

Finding a Sparse LU Decomposition

We usually have a choice of non-zero elements to use next. We can use this freedom to try to make $\mathbf{L}$ and $\mathbf{U}$ as sparse as possible.

One strategy is the *minimal column* heuristic:

> Pick a non-zero element in row i or later from a column of $\mathbf{F}$ (from i onwards) that has the minimal number of non-zeros (but which does have a non-zero at row i or later).

This minimizes the number of non-zeros that will be immediately added to $\mathbf{L}$ and $\mathbf{U}$.

The *minimal product* heuristic is more forward looking:

> Pick the non-zero element from row i, column i or later that minimizes the product of
>
> - the number of non-zeros in its row minus 1
> - the number of non-zeros in its column (from row i on) minus 1.

This minimizes the number of modifications to other rows, which often produce non-zeros that are of later significance.

The Matrix $\mathbf{A}^{-1}\mathbf{B}$ *for a Rate* $1/2$ *LDPC Code with 3 Checks per Bit,* $M = 35$

0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	#	0
0	0	0	0	#	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0
#	0	0	0	0	0	0	0	0	0	#	#	#	0	0	#	0	0	0	0	0	0	0	#	#	#	#	0	0	0	0	0	#	#	0
0	0	#	0	0	0	0	0	#	0	#	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	#	#	#	#	0	#	#	#	#	0	0	#	0	#	#	0	#	#	#	0	0	#	#	#	0	#	#	0	#	#	#
#	0	#	0	#	0	#	0	#	0	#	0	0	#	0	0	#	#	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	#	0
#	0	0	0	#	0	#	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	#	0	#	0	#	0	#	0	0	#	#	#	#	0	#	#	#	0	#	#	0	#	0	#	0	#	#	#	0	0
#	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0
#	#	0	0	#	#	0	0	#	0	0	#	0	#	0	#	#	#	0	#	#	#	0	#	0	#	0	#	0	0	0	#	#	0	#
#	0	0	0	0	#	#	#	0	0	0	0	#	#	0	0	#	#	#	0	#	#	#	0	0	#	#	0	0	0	#	#	0	0	#
0	#	0	0	0	0	0	0	#	0	0	0	0	#	0	0	0	0	#	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	#	0	0	#	#	0	#	#	0	#	#	0	#	0	0	#	#	0	#	0	0	0	0	0	0	#	#	#	0	#	#	0	0
0	0	#	0	0	#	#	#	#	0	#	#	0	#	0	#	#	0	#	#	#	#	#	#	#	0	0	0	0	0	#	0	#	0	#
0	0	#	0	0	#	#	0	0	#	#	0	#	0	0	#	0	#	#	0	0	0	0	#	#	0	#	#	0	0	#	0	#	#	0
#	#	0	0	#	#	0	0	0	#	0	#	0	#	#	#	0	0	#	#	#	#	#	#	#	#	#	#	0	0	0	#	#	0	0
0	0	#	0	0	0	#	#	#	0	0	0	#	0	0	#	#	0	#	#	0	#	0	0	0	0	#	#	0	#	#	0	0	0	0
0	0	#	0	#	#	#	#	0	#	0	#	#	0	#	#	0	#	0	#	0	#	0	0	#	#	0	#	#	#	0	#	0	0	#
#	0	0	0	#	#	#	#	0	0	#	0	0	0	#	0	#	#	0	0	#	#	0	#	0	0	#	#	#	#	0	#	#	#	#
0	#	#	#	#	#	#	#	0	0	#	0	0	#	0	0	0	#	0	#	#	#	0	#	#	#	#	0	#	#	0	0	#	#	#
#	0	#	0	#	0	0	0	#	0	#	0	0	#	0	0	#	#	0	0	#	0	0	0	0	0	0	0	0	0	0	#	#	#	0
0	0	#	0	0	#	#	0	0	#	#	0	0	0	#	0	0	#	0	#	0	0	#	#	#	0	#	#	0	#	0	0	#	#	0
0	0	#	0	#	0	#	#	#	0	0	#	#	0	0	#	#	0	#	#	0	#	#	0	0	#	#	#	#	#	#	#	0	0	0
#	0	#	0	0	#	0	0	#	0	#	0	#	0	0	0	0	#	0	#	0	0	0	#	#	#	#	#	#	0	#	#	0	#	0
#	0	0	0	0	0	0	0	0	#	#	#	#	0	0	#	0	0	0	0	#	0	0	#	#	#	#	0	0	0	0	0	#	#	0
0	0	0	#	0	#	0	0	0	#	#	0	0	0	0	#	#	#	0	#	#	#	0	#	#	#	0	#	#	#	#	0	#	#	0
0	0	#	0	0	0	0	0	#	0	#	#	#	0	0	0	0	0	0	0	#	0	0	0	#	0	#	#	0	#	#	#	0	#	#
#	#	0	0	#	#	0	0	0	0	0	#	0	#	0	#	#	#	0	#	#	#	#	#	#	#	#	#	0	0	0	#	#	0	#
0	0	0	#	0	0	0	0	0	0	#	#	0	0	#	0	0	0	0	0	0	0	0	0	0	#	#	0	#	0	#	0	#	0	0
0	0	#	0	0	#	#	0	#	#	0	#	0	0	0	#	0	#	#	0	#	0	0	0	0	0	0	#	0	#	0	#	#	0	0
0	0	#	0	0	0	#	0	#	0	#	#	#	#	#	#	#	#	#	#	0	0	#	0	#	0	#	#	#	#	#	#	0	#	0
0	#	#	0	0	0	#	#	0	#	0	0	#	#	0	#	#	0	0	0	0	0	0	0	0	0	#	#	0	0	#	0	0	0	0
#	0	0	0	0	#	#	#	0	0	0	0	#	#	0	#	#	#	#	0	#	#	#	0	0	#	#	#	0	0	0	#	#	0	#
0	#	0	0	#	#	#	#	0	#	#	#	#	0	0	#	0	#	#	0	#	0	#	#	0	#	#	#	0	#	#	0	#	#	#

The Matrix $\mathbf{A}^{-1}$ *for This Code.*

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#	#	0	#	#	0	#	0	#	0	#	#	0	#	0	0	0	0	0	#	0	0	#	0	0	#	0	0	0	0	0	0	0	#	#
0	#	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	#	#	#	0	#	0	0	0	#	#	0	0	#	#	0	#	#	#	0	0	0	#	0	#	0	0	#	#	#	#	0	0	#	#
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#	#	0	#	#	#	0	#	#	#	0	0	0	0	0	#	#	#	#	0	#	0	#	#	0	#	0	0	0	#	#	0	0	0	0
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0	#	0	0	0	#	#	#	0	#	#	#	0	#	#	#	#	#	0	#	#	0	#	#	#	0	0	0	0	#	#	0	0	#	#
#	0	#	#	#	0	0	#	0	#	0	#	#	#	0	#	0	0	0	0	0	0	#	#	#	#	0	#	0	#	#	#	#	0	#
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0	#	0	#	0	#	0	#	#	#	0	#	#	#	#	0	0	0	0	#	#	#	#	0	0	#	0	0	#	#	0	0	0	#	#
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#	0	#	#	0	#	#	#	0	#	0	#	#	0	0	#	0	0	#	#	#	#	0	0	0	0	0	#	#	0	0	0	0	0	0
#	0	#	#	0	0	#	#	#	#	0	#	0	#	0	#	#	#	0	0	0	#	0	0	#	0	0	#	#	0	0	#	#	0	#
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#	0	#	#	0	0	#	#	0	0	0	#	0	0	0	#	0	0	#	#	#	#	0	0	0	0	0	#	#	0	0	0	0	0	0
0	#	#	#	#	#	0	0	#	0	0	#	#	#	0	0	#	#	#	0	#	0	0	0	#	0	0	#	0	0	0	#	#	0	#
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0	0	#	#	#	0	0	#	0	#	0	#	#	#	#	#	#	0	#	#	0	#	0	0	0	#	#	#	0	#	#	#	0	0	0
0	#	0	#	0	0	0	#	#	#	0	#	#	#	#	0	0	0	0	#	#	#	#	0	0	#	0	0	#	#	0	0	0	#	#
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0	0	0	0	#	0	#	0	#	0	0	0	#	0	#	0	#	#	0	0	#	#	#	#	0	#	0	0	#	#	#	0	0	0	0
0	0	#	#	0	0	0	0	0	0	0	#	0	0	0	#	0	0	#	#	#	0	0	0	0	0	0	#	#	0	0	0	0	0	0
#	#	0	0	#	#	0	#	#	#	0	0	0	0	0	#	#	#	#	0	#	0	#	#	0	#	0	0	0	#	#	0	0	0	0
#	#	0	#	#	0	#	0	#	0	0	#	0	#	0	0	0	0	0	#	0	0	#	0	0	#	0	0	0	0	0	0	0	#	#

The $\mathbf{L}$ *Matrix for This Code, As Found With the Minimal Product Heuristic*

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 # 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 # 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 # 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 # 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 # 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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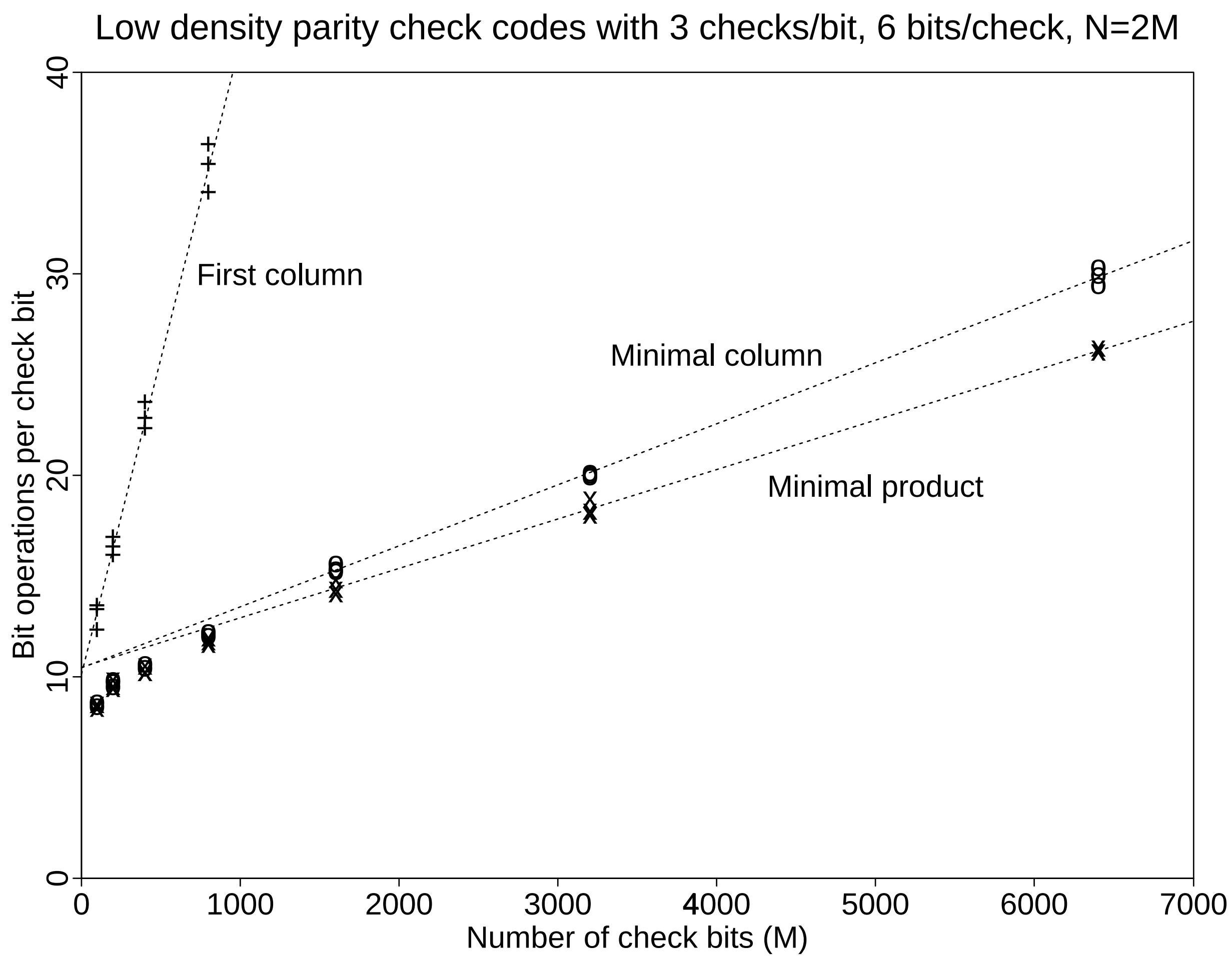
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 # 0 0 0 0

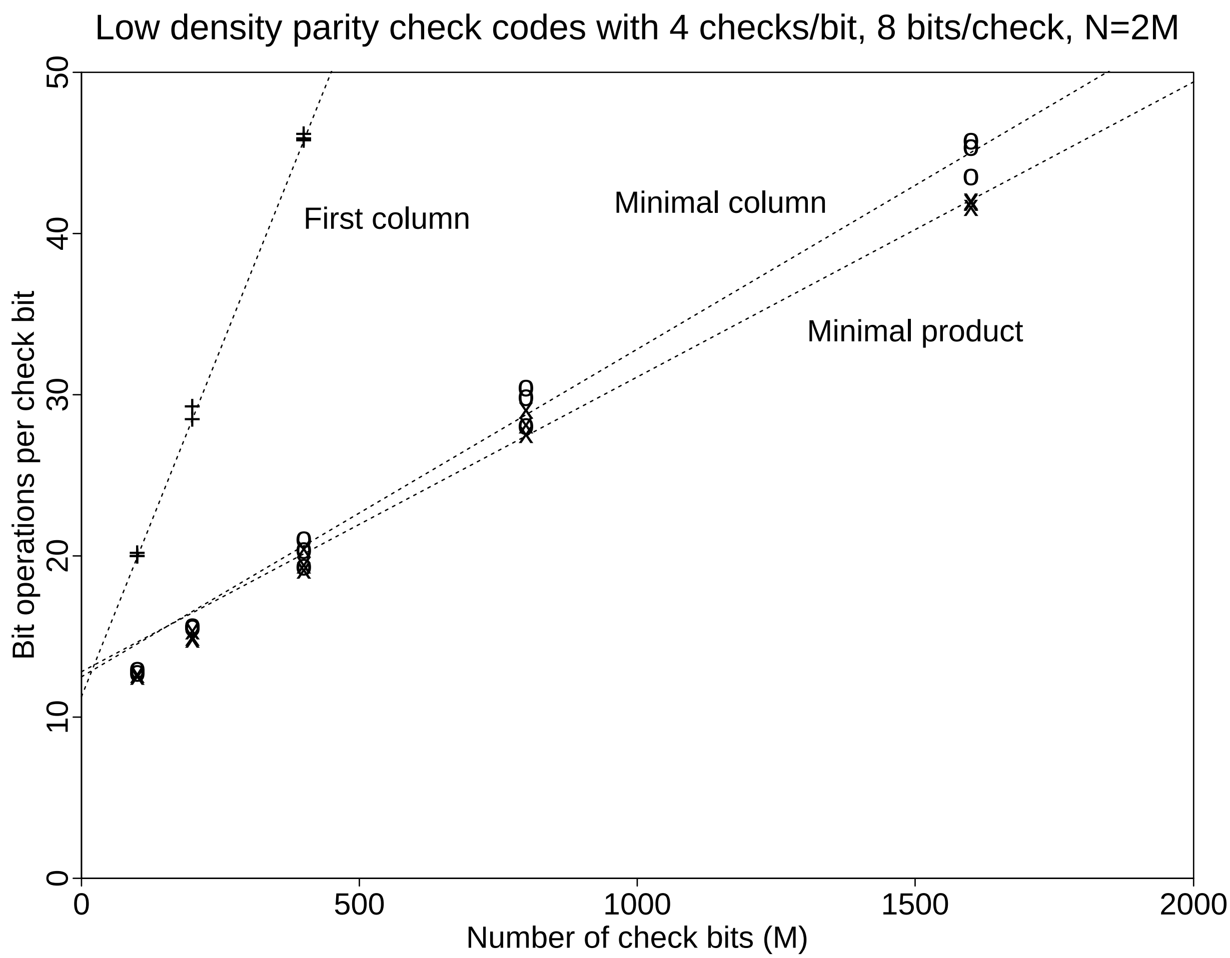
The **U** *Matrix for This Code,*
As Found With the Minimal Product Heuristic

#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0
0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#
0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	#	0	0	0	0	0
0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0
0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	#	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	#	0	0	#	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	#	0	0	0	0	0	0	#	0	#	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	#	0	0	0	0	#	#	0	#	#	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	#	0	#	0	0	0	0	#
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	0	0	0	0	#	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	#	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	#	#	0	0	0	0	#	0	#
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	#	#	0	#	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	#	#	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	#	0	#	#
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	0	0	#
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0	#	#
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	#	#	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	#

Results on Codes With 3 Checks per Bit



Results on Codes With 4 Checks per Bit



Summary

- A fairly standard LU decomposition approach can greatly reduce the number of bit operations for encoding low density parity check codes.

- For standard LDPC codes, the number of operations per check still grows linearly with block size, but at a slow rate. Hence encoding still takes time proportional to N^2, but with a small constant factor.

- For moderate block sizes, dense matrix operations can still be faster, especially in software, due to the parallelism possible by operating on 32 bits at a time.

- The process of forward substitution resembles that of encoding a recursive convolutional code.